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A Numerical Example of Optimal Bang-Bang Controls

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SUMMARY
The problem of computing optimal bang-bang controls for a nonlinear control system is discussed. It is shown that the method of replacing the optimal bang-bang control problem by a parameter optimization problem leads to an efficient algorithm. The problem of bringing a rotating rigid body to rest in minimum time is used to illustrate the theory. In this example, the parameter optimization problem reduces to a one-dimensional search.

1. Introduction

Many attemps have been made in recent years to develop systematic computational procedures for the solution of nonlinear optimal control problems. The present paper deals with a special class of such problems, namely those for which the optimal control is known to be of bang-bang type. Sufficient conditions for the optimal control to be bang-bang are that each control variable is (a) bounded above and below, (b) appears linearly in the state equations, and (c) appears linearly or not at all in the performance index. These conditions are quite common in practice, for example in aerospace studies.

Bryson, Denham and Dreyfus [1] and Denham and Bryson [2] extended the gradient method of Kelley [3] to the case when inequality constraints are present in the control and state variables. In particular, they described a procedure for the numerical determination of optimal bang-bang controls. The basic idea of their method is to replace the control problem by a parameter optimization problem, the unknown parameters being the switching times for the controls.

In subsequent sections, we shall use parameter optimization in conjunction with the Pontryagin Minimum Principle [4] to solve the problem of bringing a rotating rigid body to rest in minimum time, when the couples that may be applied to the body are bounded. It will be seen that the resulting iterative process is relatively simple and that the convergence in the numerical example (section 4) is rapid.

2. General Theory

This section follows closely the relevant part of reference [2]. The problem to be considered may be stated as follows. Suppose that we are given a control system with state equations

$$\dot{\boldsymbol{x}} = \boldsymbol{f}[\boldsymbol{x}(t), \boldsymbol{u}(t), t], \tag{1}$$

where $\boldsymbol{x} = [x_1, x_2, x_3]$. The components u_i of the control vector $\boldsymbol{u}$ are bounded above and below:

$$u_{il} \leqq u_i \leqq u_{iu}, \qquad i = 1, 2, 3. \tag{2}$$

The initial time t_0 and the initial state vector $\boldsymbol{x}(t_0)$ are given. At the final time t_f, we require

$$\Omega[\boldsymbol{x}(t_f), t_f] = 0 \tag{3}$$

and

$$\boldsymbol{\psi}[\boldsymbol{x}(t_f), t_f] = \boldsymbol{0}, \tag{4}$$

where Ω and the components of $\boldsymbol{\psi} = [\psi_1, \psi_2]$ are given functions of $\boldsymbol{x}(t)$ and t. The problem is to find a control vector $\boldsymbol{u}$ that transfers the state vector from $\boldsymbol{x}(t_0)$ to $\boldsymbol{x}(t_f)$ in minimum time $t_f - t_0$, in such a way that (1)–(4) are all satisfied.

The reason for separating the terminal constraints into two parts, (3) and (4), is that in the iterative process to be developed it is convenient to use (3) to determine the final time t_f. Ideally, if $\boldsymbol{x}^*(t)$ is the optimal trajectory, then the function $\Omega[\boldsymbol{x}^*(t), t]$ should not vanish in the open interval $0 < t < t_f$. However, the method is easily modified to include the case when $\Omega[\boldsymbol{x}^*(t), t]$ has zeroes in this interval. If it is known that $t = t_f$ corresponds to the 2nd, 3rd, ... zero of $\Omega[\boldsymbol{x}(t), t]$ in the interval $0 < t \leqq t_f$, then equation (3) may be replaced by

"t_f is the value of t for which $\Omega[\boldsymbol{x}(t), t]$ vanishes for the 2nd, 3rd, ... time".

If it is not known which zero of $\Omega[\boldsymbol{x}(t), t]$ corresponds to t_f, then we define $t_{f1}, t_{f2}, \ldots$, corresponding to the 1st, 2nd, ... zero of $\Omega[\boldsymbol{x}(t), t]$ in the interval $0 < t \leqq t_f$, solve the optimization problems using each of these values of t_f in turn, and compare the final results to give the optimal solution.

Assuming that $\boldsymbol{u}$ appears linearly in eqs. (1), the Pontryagin Minimum Principle applied to the above problem leads immediately to the result that the optimal control is bang-bang. We therefore choose a nominal bang-bang control which satisfies (2) and whose components u_i, $i = 1, 2, 3$, have switching times t_i, respectively; the problem is thus reduced to finding the optimal switching times t_i. More generally, we could assume any number of switching times for each component of the control. The present assumption is the simplest possible, but is adequate for the example of sections 3 and 4. It is pointed out by Denham and Bryson [2] that using more switching times than necessary in the nominal control will, in general, lead to the optimal control, because two switching times can become equal in the limit; on the other hand, no additional switching times can be added by the present technique.

Substituting the nominal control in equations (1), we can solve these equations forwards in time and determine t_f from equation (3). Knowing t_f, it is possible to find out how closely the remaining terminal conditions (4) are satisfied. If they are satisfied within the required accuracy, then we have already found the optimal bang-bang control. If they are not, as is usually the case, then we produce small perturbations $\boldsymbol{\delta x}$ in the nominal state vector by making small changes dt_i in the switching times t_i of the nominal control. We consider the corresponding small perturbations $d\Omega$, $\boldsymbol{d\psi}$ of the terminal constraint functions Ω, $\boldsymbol{\psi}$. We can express $d\Omega$ in the form

$$d\Omega = \delta\Omega + \dot{\Omega}\, dt_f\,, \tag{5}$$

where dt_f is a small perturbation in final time t_f, and $\delta\Omega$ is that part of $d\Omega$ which is independent of t_f. Similarly, we can express $\boldsymbol{d\psi}$ in the form

$$\boldsymbol{d\psi} = \boldsymbol{\delta\psi} + \dot{\boldsymbol{\psi}}\, dt_f\,. \tag{6}$$

Next, we find a matrix of influence functions $\boldsymbol{\lambda}_\psi^f$ such that

$$\boldsymbol{\delta\psi} = (\boldsymbol{\lambda}_\psi^f)^T \boldsymbol{\delta x}^f\,, \tag{7}$$

where all the quantities are evaluated at $t = t_f$. In the present problem, λ_ψ^f is a (3×2) matrix. We shall now prove that when equation (4) is not satisfied, the required change $\boldsymbol{\delta\psi}$ in $\boldsymbol{\psi}$ is related to a change $\boldsymbol{\delta u}$ in $\boldsymbol{u}$ by the relation

$$\boldsymbol{\delta\psi} = \int_{t_0}^{t_f} \boldsymbol{\lambda}_\psi^T \boldsymbol{G}\, \boldsymbol{\delta u}\, dt\,, \tag{8}$$

where $\boldsymbol{\lambda}_\psi$ is the solution of the homogeneous linear differential equations

$$\dot{\boldsymbol{\lambda}}_\psi = -\boldsymbol{F}^T \boldsymbol{\lambda}_\psi\,, \tag{9}$$

with boundary conditions

$$\lambda_{\psi ik}^f = \left(\frac{\partial \psi_k}{\partial x_i}\right)_{t = t_f}, \qquad i = 1, 2, 3\;; \quad k = 1, 2\,, \tag{10}$$

and

$$F_{ij} = \frac{\partial f_i}{\partial x_j}, \quad G_{ij} = \frac{\partial f_i}{\partial u_j}, \qquad j = 1, 2, 3 . \tag{11}$$

Note that equations (9) are the usual adjoint equations as used in the Pontryagin Minimum Principle, though with different boundary conditions.

If $\boldsymbol{\delta x}$ is a small perturbation of the state vector $\boldsymbol{x}$, then, from equations (1) and (11), it must satisfy

$$\boldsymbol{\delta \dot{x}} = \boldsymbol{F}\,\boldsymbol{\delta x} + \boldsymbol{G}\,\boldsymbol{\delta u} . \tag{12}$$

Premultiply equation (12) by $\boldsymbol{\lambda}_\psi^T$, postmultiply the transpose of equation (9) by $\boldsymbol{\delta x}$, and add the resulting equations to give

$$\frac{d}{dt}[\boldsymbol{\lambda}_\psi^T \boldsymbol{\delta x}] = \boldsymbol{\lambda}_\psi^T \boldsymbol{G}\,\boldsymbol{\delta u} , \tag{13}$$

i.e.

$$(\boldsymbol{\lambda}_\psi^f)^T \boldsymbol{\delta x}^f = \int_{t_0}^{t_f} \boldsymbol{\lambda}_\psi^T \boldsymbol{G}\,\boldsymbol{\delta u}\,dt , \tag{14}$$

since $\boldsymbol{\delta x}(t_0) = \boldsymbol{0}$ by hypothesis. Equations (7) and (14) give equation (8). It remains to show that $\boldsymbol{\lambda}_\psi$ must satisfy the boundary conditions (10), but this result follows immediately on writing the left-hand side of equation (7) in the form

$$\boldsymbol{\delta\psi} = \sum_{i=1}^{3} \frac{\partial \boldsymbol{\psi}^f}{\partial x_i} \delta x_i^f \tag{15}$$

and identifying the coefficients of the δx_i^f on the right-hand sides of equations (7) and (15).

By reasoning similar to that which led to equations (7), (9) and (10), we can obtain a vector of influence functions $\boldsymbol{\lambda}_\Omega$ such that

$$\delta\Omega = (\boldsymbol{\lambda}_\Omega^f)^T \boldsymbol{\delta x}^f , \tag{16}$$

where

$$\dot{\boldsymbol{\lambda}}_\Omega = -\boldsymbol{F}^T \boldsymbol{\lambda}_\Omega , \tag{17}$$

with

$$\boldsymbol{\lambda}_\Omega^f = \boldsymbol{\nabla}\Omega(t_f) , \tag{18}$$

and corresponding to equation (8) we have

$$\delta\Omega = \int_{t_0}^{t_f} \boldsymbol{\lambda}_\Omega^T \boldsymbol{G}\,\boldsymbol{\delta u}\,dt . \tag{19}$$

Since $d\Omega = 0$, equation (5) gives

$$dt_f = -\delta\Omega/\dot{\Omega}$$

$$= -(1/\dot{\Omega}) \int_{t_0}^{t_f} \boldsymbol{\lambda}_\Omega^T \boldsymbol{G}\,\boldsymbol{\delta u}\,dt \tag{20}$$

$$= -\int_{t_0}^{t_f} \boldsymbol{\lambda}_{\rho\Omega}^T \boldsymbol{G}\,\boldsymbol{\delta u}\,dt , \tag{21}$$

where

$$\boldsymbol{\lambda}_{\rho\Omega} = \boldsymbol{\lambda}_\Omega/\dot{\Omega} . \tag{22}$$

Suppose that $\Delta\psi_1^f$ and $\Delta\psi_2^f$ are the values of ψ_1 and ψ_2, respectively, on the nominal trajectory at $t = t_f$. The changes in ψ_1 and ψ_2 that are required in order to satisfy equation (4) are therefore $-\Delta\psi_1^f$ and $-\Delta\psi_2^f$, respectively. Thus, using equations (6), (8) and (20), we can write

$$\boldsymbol{d\psi} = \int_{t_0}^{t_f} \boldsymbol{\lambda}_\psi^T \boldsymbol{G}\,\boldsymbol{\delta u}\,dt - (\dot{\boldsymbol{\psi}}/\dot{\Omega}) \int_{t_0}^{t_f} \boldsymbol{\lambda}_\Omega^T \boldsymbol{G}\,\boldsymbol{\delta u}\,dt = -[\Delta\psi_1^f, \Delta\psi_2^f] , \tag{23}$$

or $$d\psi = \int_{t_0}^{t_f} \lambda_{\psi\Omega}^{\mathrm{T}} G\,\delta u\,dt = -[\Delta\psi_1^f, \Delta\psi_2^f]\,, \tag{24}$$

where

$$\lambda_{\psi\Omega} = \lambda_\psi - (\lambda_\Omega \dot{\psi}^{\mathrm{T}})/\dot{\Omega}\,. \tag{25}$$

The optimization problem that we shall solve is: Find δu to minimize dt_f, given by equation (21), subject to the constraints (24). The next step is to express the integrals involving δu in terms of the incremental switching times dt_1, dt_2, dt_3. One iteration of the method is complete when these increments have been determined. However, to avoid a multiplicity of cases, we leave the general theory at this point and consider a specific example.

3. Rotational Motion of a Rigid Body

The general equations of rotational motion of a rigid body may be written in the well-known Euler form

$$\left.\begin{aligned} A\dot{p} - (B-C)qr &= L\,, \\ B\dot{q} - (C-A)rp &= M\,, \\ C\dot{r} - (A-B)pq &= N\,, \end{aligned}\right\} \tag{26}$$

where A, B, C are the principal moments of inertia of the body at its centre of mass. Referred to the principal axes of the body at its centre of mass, p, q, r are the components of angular velocity and L, M, N are the components of the applied couple.

By simple changes of variables and parameters, equations (26) can be expressed in the standard form for state equations:

$$\left.\begin{aligned} \dot{x}_1 &= ax_2x_3 + u_1 \equiv f_1(x, u)\,, \\ \dot{x}_2 &= bx_3x_1 + u_2 \equiv f_2(x, u)\,, \\ \dot{x}_3 &= cx_1x_2 + u_3 \equiv f_3(x, u)\,. \end{aligned}\right\} \tag{27}$$

We assume that the values of the x_i at the initial time $t = t_0$ are given, and that the u_i satisfy the constraints (2).

The problem we shall solve is that of bringing the rotating rigid body to rest in minimum time. The solution of this problem by the method of "backing out of the terminal state" was discussed in reference [5]. The present method appears to be superior, since the resulting iterative process involves much less trial and error.

The terminal constraints on the state variables are simply

$$x_i(t_f) = 0\,, \qquad i = 1, 2, 3\,. \tag{28}$$

We use the first of these to determine t_f on the nominal trajectory. Thus, in the notation of the previous section

$$\Omega[x(t_f), t_f] \equiv x_1(t_f) = 0\,, \tag{29}$$

$$\psi[x(t_f), t_f] \equiv [\psi_1^f, \psi_2^f] \equiv [x_2(t_f), x_3(t_f)] = 0\,. \tag{30}$$

From equations (10), (11), (27), (29) and (30), we find

$$F = \begin{pmatrix} 0 & ax_3 & ax_2 \\ bx_3 & 0 & bx_1 \\ cx_2 & cx_1 & 0 \end{pmatrix}, \tag{31}$$

$$\lambda_\psi^f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{32}$$

$$G = I . \tag{33}$$

Next, we convert the integrals involving δu into expressions involving the dt_i. Consider first equation (21), which, because of equation (33), now becomes

$$dt_f = -\int_{t_0}^{t_f} \lambda_{\rho\Omega}^T \delta u \, dt . \tag{34}$$

Suppose that at time t_i ($i=1, 2, 3$) the control u_i switches from its minimum value to its maximum value if $x_i(t_0) > 0$, and from its maximum value to its minimum value if $x_i(t_0) < 0$. Suppose also that dt_i is so small that the ith component of $\lambda_{\rho\Omega}$ may be regarded as constant between t_i and $t_i + dt_i$. Then we find

$$dt_f = -\int_{t_0}^{t_f} \lambda_{\rho\Omega}^T \delta u \, dt = \sum_{i=1}^{3} \operatorname{sgn}[x_i(t_0)](u_{iu} - u_{il})\lambda_{\rho\Omega i}(t_i) dt_i , \tag{35}$$

where $\lambda_{\rho\Omega i}(t_i)$ is the ith component of $\lambda_{\rho\Omega}$ evaluated at $t = t_i$. Similarly, from equation (24), we find

$$d\psi = -\begin{pmatrix} \lambda_{\psi\Omega 11}(t_1) \lambda_{\psi\Omega 21}(t_2) \lambda_{\psi\Omega 31}(t_3) \\ \lambda_{\psi\Omega 12}(t_1) \lambda_{\psi\Omega 22}(t_2) \lambda_{\psi\Omega 32}(t_3) \end{pmatrix} \begin{pmatrix} \operatorname{sgn}[x_1(t_0)](u_{1u} - u_{1l}) dt_1 \\ \operatorname{sgn}[x_2(t_0)](u_{2u} - u_{2l}) dt_2 \\ \operatorname{sgn}[x_3(t_0)](u_{3u} - u_{3l}) dt_3 \end{pmatrix} = -\begin{pmatrix} \Delta\psi_1^f \\ \Delta\psi_2^f \end{pmatrix} . \tag{36}$$

The optimization problem now becomes: Given the switching times t_i, find the dt_i to minimize dt_f of equation (35), subject to the constraints (36).

To take a definite case, let us assume

$$x_1(t_0) < 0 , \quad x_2(t_0) > 0 , \quad x_3(t_0) > 0 ,$$

together with

$$|u_1| \leqq u_{1m} , \quad |u_3| \leqq u_{3m} .$$

We still assume

$$u_{2l} \leqq u_2 \leqq u_{2u} .$$

The constraints on u_1, u_2, u_3 represent the usual restrictions on the movements of the ailerons, elevator and rudder, respectively, on an orthodox aircraft.

To simplify the notation, we write

$$T_i = dt_i .$$

The optimization problem of equations (35) and (36) may then be written: Find T_i, $i=1, 2, 3$, to minimise

$$\left. \begin{aligned} & dt_f = \sum_{i=1}^{3} c_i T_i \\ & \text{subject to the constraints} \\ & \sum_{i=1}^{3} a_{ij} T_i = b_j , \qquad j = 1, 2 , \end{aligned} \right\} \tag{37}$$

where

$$\left. \begin{aligned} c_1 &= -2u_{1m}\lambda_{\rho\Omega 1}, & c_2 &= (u_{2u} - u_{2l})\lambda_{\rho\Omega 2}, & c_3 &= 2u_{3m}\lambda_{\rho\Omega 3} , \\ a_{11} &= -2u_{1m}\lambda_{\psi\Omega 11}, & a_{21} &= (u_{2u} - u_{2l})\lambda_{\psi\Omega 21}, & a_{31} &= 2u_{3m}\lambda_{\psi\Omega 31} , \\ a_{12} &= -2u_{1m}\lambda_{\psi\Omega 12}, & a_{22} &= (u_{2u} - u_{2l})\lambda_{\psi\Omega 22}, & a_{32} &= 2u_{3m}\lambda_{\psi\Omega 32} , \end{aligned} \right\} \tag{38}$$

$$b_1 = \Delta x_2(t_f), \qquad b_2 = \Delta x_3(t_f) ,$$

and the λ's are evaluated at the appropriate switching times, as indicated in equations (35) and (36).

Problem (37) is solved by first eliminating T_2 and T_3 from the objective function by means of the constraint equations. We find

$$dt_f = A_1 T_1 + B_1 \,, \tag{39}$$

where

$$\left.\begin{aligned} A_1 &= c_1 + c_2 E_2 + c_3 E_3 \,, \\ B_1 &= c_2 F_2 + c_3 F_3 \,, \end{aligned}\right\} \tag{40}$$

with

$$\left.\begin{aligned} E_2 &= (a_{12}a_{31} - a_{11}a_{32})/D \,, \quad E_3 = (a_{11}a_{22} - a_{12}a_{21})/D \,, \\ F_2 &= (b_1 a_{32} - b_2 a_{31})/D \,, \quad F_3 = (b_2 a_{21} - b_1 a_{22})/D \,, \end{aligned}\right\}$$

and

$$D = a_{21}a_{32} - a_{31}a_{22} \,. \tag{41}$$

Since dt_f is to be minimised, equation (39) shows that

$$T_1 = -X \operatorname{sgn}[A_1] \,, \tag{42}$$

where $X \geqq 0$ is subject only to the condition that it must be small enough for the theory to be valid. We shall return to this point later. We next find

$$T_2 = E_2 T_1 + F_2 \,, \tag{43}$$

$$T_3 = E_3 T_1 + F_3 \,. \tag{44}$$

The computation of T_1, T_2 and T_3 completes one iteration of the method, the new switching times being $t_1 + T_1$, $t_2 + T_2$ and $t_3 + T_3$. A summary of the computational procedure follows.

3.1. *Computational Procedure*

1. Choose a nominal control, and integrate equations (27) forwards in time, stopping at time t_f defined by $x_1(t_f) = 0$. Retain the values of $\boldsymbol{x}(t)$, and find

   $$b_1 = x_2(t_f) \,, \quad b_2 = x_3(t_f) \,, \quad |\boldsymbol{x}(t_f)| \,.$$

   If $|\boldsymbol{x}(t_f)| < \varepsilon$, where ε is some pre-assigned quantity, then the calculation terminates. Otherwise:
2. Evaluate $\dot{\Omega} = \dot{x}_1(t_f) = [ax_2x_3 + u_1]_{t=t_f}$, and

   $$\dot{\psi} = [\dot{x}_2, \dot{x}_3]_{t=t_f} = [bx_3x_1 + u_2, cx_1x_2 + u_3]_{t=t_f} \,.$$

3. Integrate equations (9) and (17) backwards in time, and find the components of $\boldsymbol{\lambda}_\psi$ and $\boldsymbol{\lambda}_\Omega$ at the appropriate switching times—see equations (35) and (36).
4. Evaluate the components of $\boldsymbol{\lambda}_{\psi\Omega}$ and $\boldsymbol{\lambda}_{\rho\Omega}$ from equations (22) and (25), respectively, at the appropriate switching times.
5. Evaluate the coefficients in equations (37); these are given in equations (38).
6. Evaluate $D, E_2, E_3, F_2, F_3, A_1$ from equations (40) and (41).
7. Evaluate T_1, T_2, T_3 from equations (42)–(44) for several values of X ($\geqq 0$), starting with $X = 0$.
8. Return to step 1 using a new nominal control with switching times $t_1 + T_1$, $t_2 + T_2$, $t_3 + T_3$.

Notes

(i) When $A_1 = 0$, the final time t_f is invariant for small changes T_1, T_2, T_3 in the switching times. For the condition $A_1 = 0$ can be written

$$\begin{vmatrix} \lambda_{\rho\Omega 1} & \lambda_{\rho\Omega 2} & \lambda_{\rho\Omega 3} \\ \lambda_{\psi\Omega 11} & \lambda_{\psi\Omega 21} & \lambda_{\psi\Omega 31} \\ \lambda_{\psi\Omega 12} & \lambda_{\psi\Omega 22} & \lambda_{\psi\Omega 32} \end{vmatrix} = 0 ,$$

which is merely a statement of the consistency of the condition $dt_f=0$ with the required terminal conditions $d\Omega=0$ and $d\boldsymbol{\psi}=\boldsymbol{0}$.

(ii) $B_1=0$ ultimately, since the terminal constraints are satisfied when $\Delta x_2(t_f)=\Delta x_3(t_f)=0$.

(iii) In step 7, it is found convenient to search for a value of X which gives the closest approximation to the required terminal conditions, *regardless* of whether this X is "sufficiently small" as required by the theory. In other words, equation (39) is used solely to determine the sign of T_1.

(iv) In common with all gradient methods, the algorithm must converge to a local optimum. In general, it cannot be guaranteed that the global optimum will be found. However, repeating the calculations with different nominal trajectories will lead to either (a) a better result, or (b) more confidence in the given result.

4. Numerical Example

The numerical example is the same as that of reference [5]. In equations (26), we take

$$A:B:C = 3:8:10 .$$

Then, equations (27) become

$$\begin{aligned} \dot{x}_1 &= -0.667 x_2 x_3 + u_1 , \\ \dot{x}_2 &= 0.875 x_3 x_1 + u_2 , \\ \dot{x}_3 &= -0.500 x_1 x_2 + u_3 . \end{aligned} \tag{45}$$

The bounds on the controls are taken to be

$$u_{1m} = 0.40 , \quad u_{2l} = -0.20 , \quad u_{2u} = 0.13 , \quad u_{3m} = 0.14 , \tag{46}$$

and are intended to be representative values for an orthodox aircraft. We take $t_0=0$, and

$$x_1(0) = -2.061 , \quad x_2(0) = 0.106 , \quad x_3(0) = 0.746 . \tag{47}$$

It is known from reference [5] that the time-optimal control will bring the system (45)–(47) to rest in about 5 sec.

We assume that at $t=0$ the controls are set in the directions which oppose the respective components of angular velocity, i.e.

$$u_1(0)>0 , \quad u_2(0)<0 , \quad u_3(0)<0 .$$

For the nominal control we choose switching times $t_1=5.5$, $t_2=0.5$, $t_3=1.5$, and integrate equations (45), with (46) and (47), forward in time until $x_1(t)=0$. All the numerical integrations were performed using a Runge–Kutta–Merson subroutine, with steps of 0.1 in t. Smaller steps would have given greater accuracy, but the method can be adequately illustrated with this step size.

From the results of these integrations, we find

$$t_f = 5.1 , \quad b_1 = 0.2911 , \quad b_2 = 0.0000 , \quad |\boldsymbol{x}(t_f)| = 0.2917 .$$

(The result for b_2 is fortuitous).

Next, we evaluate

$$\dot{\Omega} = 0.40 , \quad \boldsymbol{\psi} = [0.13, 0.14] .$$

In equations (9) and (17) for the influence functions, we reverse the time coordinate by writing $\tau=t_f-t$. These equations are integrated from $\tau=0$ to $\tau=t_f$ with initial conditions (10) and (18):

$$\begin{pmatrix} \lambda_{\psi 11} & \lambda_{\psi 12} \\ \lambda_{\psi 21} & \lambda_{\psi 22} \\ \lambda_{\psi 31} & \lambda_{\psi 32} \end{pmatrix}_{\tau=0} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$[\lambda_{\Omega 1}, \lambda_{\Omega 2}, \lambda_{\Omega 3}]_{\tau=0} = [1, 0, 0] .$$

The switching times in the reversed time coordinate are

$$\tau_1 = t_f - t_1 = -0.4 ,$$
$$\tau_2 = t_f - t_2 = 4.6 ,$$
$$\tau_3 = t_f - t_3 = 3.6 .$$

Since $\tau_1 < 0$, we replace this switching time by $\tau_1 = 0$ and find the components of $\boldsymbol{\lambda}_\psi$ and $\boldsymbol{\lambda}_\Omega$ at the appropriate switching times. Then, using equations (22) and (25), we find

$$\boldsymbol{\lambda}_{\psi\Omega} = \begin{pmatrix} -0.3250 & -0.3500 \\ -0.3182 & 0.5478 \\ -1.1485 & 0.3713 \end{pmatrix}$$

and

$$\boldsymbol{\lambda}_{\rho\Omega} = [2.5, 0.7222, -0.7982] .$$

It is now a simple matter to evaluate the coefficients on the left-hand sides of equations (38). Thence, using equations (40) and (41), we find

$$D = 0.04723 ,$$
$$E_2 = -2.479 , \quad E_3 = 1.618 ,$$
$$F_2 = 0.641 , \quad F_3 = -1.114 ,$$
$$A_1 = -2.952 .$$

From equations (42)–(44), we construct the following table, in which the switching times are given to the nearest 0.1 sec.:

TABLE 1

Search for new nominal control: switching times

$X = T_1$	T_2	T_3	$t_f + T_1$	$t_2 + T_2$	$t_3 + T_3$
0	0.6	−1.1	5.1	1.1	0.4
0.1	0.4	−1.0	5.2	0.9	0.5
0.2	0.1	−0.8	5.3	0.6	0.7
0.3	−0.1	−0.6	5.4	0.4	0.9
0.4	−0.4	−0.5	5.5	0.1	1.0
0.5	−0.6	−0.3	5.6	−0.1	1.2
0.6	−0.8	−0.1	5.7	−0.3	1.4

Note that the new switching time for u_1 is $t_f + T_1$, rather than $t_1 + T_1$, since $t_1 > t_f$ for the nominal control.

Returning to step 1 of the computational procedure, we integrate the state equations (45)–(47) forward in time for each of the cases of Table 1, using the new switching times, until $x_1(t) = 0$, giving $t = (t_f)_{\text{new}}$, in each case. A new switching time greater than $(t_f)_{\text{new}}$ may be taken equal to $(t_f)_{\text{new}}$, and a switching time less than zero may be taken as zero. We obtain the following results for the above seven cases:

TABLE 2

Determination of new nominal control

X	$(t_f)_{\text{new}}$	$\lvert x(t_f)_{\text{new}}\rvert$
0	5.4*	0.272*
0.1	5.3*	0.195*
0.2	5.1	0.103
0.3	5.0	0.051
0.4	5.0	0.068
0.5	5.0	0.005
0.6	5.0	0.068

It is clear from Table 2 that the case $X=0.5$ shows the greatest improvement on the nominal case. If we decide at this stage that the terminal constraints are satisfied sufficiently closely by $|x(t_f)_{\text{new}}|=0.005$, then the calculation terminates. Otherwise, the next step is to take the case $X=0.5$ as the new nominal case. We then have for the nominal trajectory,

$$t_f = 5.0\,, \quad t_1 = 5.0\,, \quad t_2 = 0.0\,, \quad t_3 = 1.2\,.$$

Proceeding as before, we find in succession

$$b_1 = -0.0049\,, \quad b_2 = -0.0002\,,$$
$$\dot{\Omega} = 0.40\,, \quad \dot{\psi} = [0.13, 0.14]\,.$$

Solving equations (9) and (17) for the influence functions in reversed time, with the same initial conditions as before, we find

$$\lambda_{\psi\Omega} = \begin{pmatrix} -0.3250 & -0.3500 \\ -0.4050 & 0.6201 \\ -1.4778 & 0.1866 \end{pmatrix}$$

and

$$\lambda_{\rho\Omega} = [2.5, 1.5012, 0.4110]\,.$$

From equations (40) and (41), we find

$$D = 0.0777$$
$$E_2 = -1.665\,, \quad E_3 = 1.166\,,$$
$$F_2 = -0.004\,, \quad F_3 = -0.013\,,$$
$$A_1 = -2.691\,.$$

Finally, from equations (42)–(44), we construct the following table, which corresponds to Table 1:

TABLE 3

Search for new nominal control: switching times

$X=T_1$	T_2	T_3	t_f+T_1	t_2+T_2	t_3+T_3
0	−0.0	−0.0	5.0	0.0	1.2
0.1	−0.2	0.1	5.1	−0.2	1.3
0.2	−0.3	0.2	5.2	−0.3	1.4
0.3	−0.5	0.3	5.3	−0.5	1.5

The cases $X=0$ and $X=0.2$ of Table 3 coincide with the cases $X=0.5$ and $X=0.6$, respectively, of Table 1. Corresponding to Table 2, we now have:

* Extrapolated, since $x_1(t)<0$ for all t in the interval $0 \leqq t \leqq t_f$.

TABLE 4

Determination of new nominal control

X	$(t_f)_{new}$	$\lvert x(t_f)_{new}\rvert$
0	5.0	0.005
0.1	5.0	0.025
0.2	5.0	0.068
0.3	5.0	0.111

It is clear from Table 4 that the best of the new cases is the case $X=0$, i.e. the case corresponding to the old nominal control. Thus, to the present degree of accuracy, the process terminates. The optimal switching times are

$$t_1 = 5.0\,, \quad t_2 = 0.0\,, \quad t_3 = 1.2\,,$$

and so the optimal controls are

$$\begin{aligned} u_1 &= 0.40 \quad \text{(no switching)}\,, \\ u_2 &= 0.13 \quad \text{(no switching)}\,, \\ u_3 &= -0.14 \quad (0 \leqq t < 1.2)\,, \\ &= +0.14 \quad (1.2 < t \leqq 5.0)\,. \end{aligned}$$

The minimum time is $(t_f)_{new} = 5.0$.

It is not known whether these optimal controls are unique. Theorems on the uniqueness of optimal controls have been proved for a very limited class of nonlinear problems [4]. Uniqueness can, however, be proved for norm-invariant systems. Now although the present problem is not norm-invariant, it becomes norm-invariant if conditions (2) on the control bounds are replaced by

$$\|\boldsymbol{u}\| = (u_1^2 + u_2^2 + u_3^2)^{\frac{1}{2}} \leqq m\,,$$

where m is a constant. This suggests there is a strong possibility that the above optimal controls are unique.

5. Conclusions

An algorithm for the computation of optimal bang-bang controls has been shown to converge successfully when applied to the problem of bringing a rotating rigid body to rest in minimum time. The method is based on the work of Bryson, Denham and Dreyfus, who showed how to change an optimal bang-bang control problem into a parameter optimization problem.

A novel feature of the present approach is that the analytic conditions on the smallness of the perturbations from the nominal controls are ignored; this appears to make the search for improved controls more efficient. The method is considered to be superior to that of "reversing out of the terminal state", since the only trial and error it involves is a simple one-dimensional search in each iteration.

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